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On the Wedderburn–Guttman theorem

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Abstract

Let A be a u by v matrix of rank a, and let M and N be u by g and v by g matrices, respectively, such that $M'AN$ is nonsingular. Then, $\text{rank}(A - N(M'AN)^{-1}M'A) = a - g$, where $g = \text{rank}(AN(M'AN)^{-1}M'A) = \text{rank}(M'AN)$. This is called Wedderburn–Guttman theorem. What happens if $M'AN$ is rectangular and/or singular? In this paper we investigate conditions under which the regular inverse $(M'AN)^{-1}$ can be replaced by a g-inverse $(M'AN)^{-}$ of some kind, thereby extending the Wedderburn–Guttman theorem. The resultant conditions look similar to those arising in seemingly unrelated contexts, namely Cochran's and related theorems on distributions of quadratic forms involving a normal random vector.

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1. Introduction

Let A be a u by v matrix of rank a, and let M and N be u by g and v by g matrices, respectively, such that $M'AN$ is nonsingular. Then,

$$\text{rank}(A - AN(M'AN)^{-1}M'A) = a - g, \tag{1}$$

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where $g = \text{rank}(AN(M'AN)^{-1}M'A) = \text{rank}(M'AN)$. This is called Wedderburn–Guttman theorem. It was originally established for $g = 1$ by Wedderburn [35, p. 69] but was later extended to $g > 1$ by Guttman [9]. Guttman [9] calls the case in which $g = 1$ Lagrange's theorem while referring to Wedderburn [35], and Rao [26, p. 69] also calls it Lagrange's theorem. However, there is no reference to Lagrange in [35] according to Hubert et al. [17]. It may thus be more appropriately called Wedderburn–Guttman theorem. Guttman [11] also showed the reverse of the theorem, that is, for (1) to hold the matrix to be subtracted from A must be of the form $AN(M'AN)^{-1}M'A$. The theorem has been used extensively in psychometrics [10,15,28] and in computational linear algebra [6,16] as a basis for extracting components which are known linear combinations of observed variables. Guttman [9,10] also discusses a special case in which A is nnd, and $M = N$. However, in this paper we mostly focus on the case in which A is rectangular.

What happens if $M'AN$ is rectangular and/or singular? Let M and N be u by p and v by q matrices, respectively, where p is not necessarily equal to q, or $\text{rank}(M'AN) < \min(p, q)$. In this case one may be tempted to replace $(M'AN)^{-1}$ in (1) by a g-inverse $(M'AN)^{-}$. However, $\text{rank}(AN(M'AN)^{-}M'A) \equiv g$ may not be equal to $\text{rank}(M'AN) \equiv h$ in this case, although $h \leqslant g \leqslant \min(\text{rank}(AN), \text{rank}(M'A))$. There are thus two versions of the extended Wedderburn–Guttman theorem:

$$\text{rank}(A - AN(M'AN)^{-}M'A) = a - g, \tag{2}$$

and

$$\text{rank}(A - AN(M'AN)^{-}M'A) = a - h. \tag{3}$$

Recently, Tian and Styan [34, Corollary 2.3] have shown that (3) holds unconditionally. However, (2) does not hold without some rank subtractivity (additivity) condition. In this paper we investigate a necessary and sufficient (ns) condition for (2) to hold. It turns out that this condition is also ns for $g = h$.

There is an additional aspect to the extended Wedderburn–Guttman theorem. It concerns the condition under which matrix $A - AN(M'AN)^{-}M'A$ is unique, while (2) above concerns the condition under which $\text{rank}(A - AN(M'AN)^{-}M'A)$ is unique and is equal to $a - g$. (There was no such distinction when $p = q = g = h$, since the two aspects coincide.) We refer the former as the "matrix identifiability" condition, and the latter as the "rank identifiability" condition.

Matrix $S = AN(M'AN)^{-}M'A$ can be written as

$$S = ABA, \tag{4}$$

where

$$B = N(M'AN)^{-}M'. \tag{5}$$

Then, the rank identifiability problem can be viewed as a rank additivity problem between two matrices, S and $A - S$ without assuming any specific structures on S such as (4). There are a number of ways of characterizing the rank additivity condition. It will be shown that S has to assume the form of (4) for some B based on the rank additivity condition, although B is not necessarily assumed to be of the form (5). We first present some results obtained without assuming (5), and then those that can only be obtained under (5).

2. Main results

Throughout this paper we use $\mathrm{Sp}(Z)$ and $\mathrm{Ker}(Z)$ to denote the range space and the null space of Z, respectively.

Lemma 2.1. *Let Z_1 and Z_2 be matrices of a same order, and define $Z = Z_1 + Z_2$. Then, the following statements are equivalent:*

(i) $\mathrm{rank}(Z) = \mathrm{rank}(Z_1) + \mathrm{rank}(Z_2)$.
(ii) $Z_1 Z^- Z_1 = Z_1$ *for any g-inverse of* Z^-.
(iii) $Z_1 Z^- Z_2 = 0$ *for any g-inverse* Z^-.
(iv) $\mathrm{Sp}(Z_1) \cap \mathrm{Sp}(Z_2) = \{0\}$, *and* $\mathrm{Sp}(Z_1') \cap \mathrm{Sp}(Z_2') = \{0\}$.
(v) $\mathrm{Sp}(Z_1) \cap \mathrm{Sp}(Z_2) = \{0\}$, *and* $\mathrm{Sp}([Z_1', Z_2']') = \mathrm{Sp}(Z')$.
(vi) $\{Z^-\} \subset \{Z_1^-\}$, *where* $\{Z^-\}$ *indicates the set of all g-inverses of* Z.
(vii) $Z_1^- Z_1 = Z_1^- Z$ (*i.e.*, $Z_1^- Z_2 = 0$) *for some* Z_1^-, *and* $Z_1 Z_1^- = Z Z_1^-$ (*i.e.*, $Z_2 Z_1^- = 0$) *for some* Z_1^-. (Z_1^-*'s in the two equations could be distinct.*)

Remarks on Lemma 2.1. Note that by symmetry Z_1 can be replaced by Z_2, or Z_1 and Z_2 can be interchanged in some of the statements above. Equivalence between (i) and (ii) has been shown by Marsaglia and Styan [20,21, (7.9) of Theorem 17] and by Mitra [23, Lemma 2.6]. That (ii) implies (iii) has been pointed out by Mitra [23, Lemma 2.7]. The reverse can be shown as follows. According to Rao and Mitra [27, Lemma 2.2.4 (iii)], (iii) implies $\mathrm{Sp}(Z_2) \subset \mathrm{Sp}(Z)$, so that $ZZ^-Z_2 = Z_2$, which leads to $Z_2 Z^- Z_2 = Z_2$ and (ii).

Equivalence between (i) and (iv) has been pointed out by Marsaglia and Styan [21], and by Mitra [23, Lemma 2.1]. Equivalence between $\mathrm{Sp}(Z_1) \cap \mathrm{Sp}(Z_2) = \{0\}$ and $\mathrm{Sp}([Z_1, Z_2]) = \mathrm{Sp}(Z)$ has been shown by Marsaglia and Styan [21, (4.13) and (4.14)], establishing the equivalence between (iv) and (v). Obviously, the same relation holds among Z_1', Z_2', and Z'.

Equivalence between (i) and (vi) has been noted by Mitra [23, Lemma 7.2], [24, Lemma 1.1]. See also Mitra [24, Theorem 2.2], which showed the equivalence between (vi) and (vii), and Baksalary and Hauke [2, (1.2)].

The three matrices satisfying Condition (i) are said to satisfy the minus partial order [12,14], which is written as $Z_1 \overset{-}{<} Z$, and $Z_2 \overset{-}{<} Z$. Two matrices, Z_1 and Z_2, are said to be weakly bi-complementary if Condition (iv) above holds ([36]; see also [18]). Two matrices, Z_1 and Z_2, are said to be parallel summable if $Z_1(Z_1 + Z_2)^- Z_2$ is invariant over the choice of $Z^- = (Z_1 + Z_2)^-$ [27]. Matrices Z_1 and Z_2 in Condition (iii) clearly satisfy this condition.

The condition under Lemma 2.1 implies $\mathrm{Sp}(Z_1)$, $\mathrm{Sp}(Z_2) \subset \mathrm{Sp}(Z)$, and $\mathrm{Sp}(Z_1')$, $\mathrm{Sp}(Z_2') \subset \mathrm{Sp}(Z')$, which in turn imply that both Z_1 and Z_2 can be expressed in the form of ABA for some B as in (4). We now assume this form for Z_1, i.e., $Z_1 = ABA = AB_1A$, and $Z_2 = A - ABA = A(A^- - B)A = AB_2A$.

Theorem 2.1 (Condition A). *Let A and B be u by v and v by u matrices, respectively. Then, the following statements are equivalent*:

(i) $ABABA = ABA$.
(ii) $ABAA^-ABA = ABA$ (*i.e.*, $A^- \in \{(ABA)^-\}$).
(iii) $(A - ABA)A^-(A - ABA) = A - ABA$ (*i.e.*, $A^- \in \{(A - ABA)^-\}$).
(iv) $ABAA^-$ *is the projector onto* $\mathrm{Sp}(ABA)$ *along* $\mathrm{Ker}(ABAA^-)$.
(v) A^-ABA *is the projector onto* $\mathrm{Sp}(A^-ABA)$ *along* $\mathrm{Ker}(ABA)$.
(vi) $\mathrm{rank}(A) = \mathrm{rank}(ABA) + \mathrm{rank}(A - ABA)$.
(vii) $ABABABA = ABABA$ *and* $\mathrm{rank}(ABA) = \mathrm{rank}(ABABA)$.
(viii) $\mathrm{tr}(AB)^2 = \mathrm{tr}(AB) = h$ *and* $\mathrm{rank}(ABA) = \mathrm{rank}(ABABA)$, *where h is the number of nonzero eigenvalues of AB which are all real*.
(ix) $\mathrm{tr}(AB)^2 = \mathrm{tr}(AB)^3 = \mathrm{tr}(AB)^4$, $\mathrm{rank}(ABA) = \mathrm{rank}(ABABA)$, *and AB has only real eigenvalues*.

Proof. Equivalences among the first six propositions follow immediately from Lemma 2.1 by setting $Z_1 = ABA$, and $Z_2 = A - ABA$.

That (i) implies (vii) is obvious. Conversely, $\mathrm{rank}(ABA) = \mathrm{rank}(ABABA)$ implies $ABA = WABABA$ for some W, but $WABABA = WABABABA = ABABA$.

That (i) implies (viii) is trivial by noting that $ABAA^-$ is idempotent under (i), and $\mathrm{tr}(AB) = \mathrm{tr}(ABAA^-)$ and $\mathrm{tr}(AB)^2 = \mathrm{tr}(ABAA^-)^2$. To show the converse, let $\lambda_k (k = 1, \ldots, h)$ be nonzero eigenvalues of AB. Then, $\mathrm{tr}(AB)^2 = \mathrm{tr}(AB) = h$ implies $\sum_{k=1}^{h}(\lambda_k - 1)^2 = 0$. Since by assumption AB has only real eigenvalues, $\lambda_k = 1$ for $k = 1, \ldots, h$. Note that AB and $ABAA^-$ have the same set of eigenvalues. Consequently, $ABAA^-$ has only unit and/or zero eigenvalues. Furthermore, $\mathrm{rank}(ABA) = \mathrm{rank}(ABABA)$ implies that $ABAA^-$ is semi-simple (i.e., $\mathrm{rank}(ABAA^-) = \mathrm{rank}(ABA) = \mathrm{rank}(ABABA) = \mathrm{rank}(ABAA^-)^2$), so that $ABAA^-$ is idempotent, from which (i) follows by way of (iv).

That (i) implies (ix) is again trivial. The converse can be proven as follows. Let $\lambda_k (k = 1, \ldots, u)$ be eigenvalues of AB. Then, $\mathrm{tr}(AB)^2 = \mathrm{tr}(AB)^3 = \mathrm{tr}(AB)^4$ implies $\sum_{k=1}^{u} \lambda_k^2(1 - \lambda_k)^2 = 0$, and since by assumption AB has only real eigenvalues, they are all zero or unity. The number of unit eigenvalues is equal to $\mathrm{tr}(AB)$. The rest of the proof follows a line similar to the above. □

Note 2.1. Condition A implies that $\mathrm{rank}(ABA) = \mathrm{rank}(AB)^2 = \mathrm{rank}(BA)^2 = \mathrm{rank}(ABABA)$, which in turn is equal to $\mathrm{tr}(AB)^2 = \mathrm{tr}(AB) = \mathrm{tr}(BA) = \mathrm{tr}(BA)^2$. By (vi), $\mathrm{rank}(A - ABA) = \mathrm{rank}(A) - \mathrm{rank}(ABA)$ which in turn is equal to $\mathrm{rank}(A) - \mathrm{tr}(AB)$, which is unique if and only if $\mathrm{tr}(AB)$ is unique. If $\mathrm{rank}(A) =$

$\operatorname{rank}(ABA)$ additionally in Condition A, $\operatorname{rank}(A - ABA) = 0$, which implies $A = ABA$, that is, $B \in \{A^-\}$.

Cline and Funderlic [7] gives a general expression for $\operatorname{rank}(A - ABA)$ that holds without any additional condition. They also note the equivalence between (ii) and (vi) in their Corollary 3.2. They further state in their Corollary 3.3 that under the representation of S in (4), (vi) is equivalent to $BAB = B$ (i.e., $A \in \{B^-\}$). However, the latter condition is equivalent to our Condition D (Lemma 2.4 below), which is stronger than Condition A. In fact, it is even stronger than Condition B1 or B2 (AB or BA being idempotent). Cline and Funderlic's conditions given in their (3.16), (3.22), (3.23) and (3.24) are similar.

Condition F (Lemma 2.6) to be discussed later may be characterized as the condition in which $\operatorname{rank}(A) = \operatorname{rank}(ABA)$ holds additionally in Condition A, and $B = N(M'AN)^-M'$. In this case, $\operatorname{rank}(ABA) = \operatorname{rank}(M'AN)$, and $\operatorname{rank}(A - ABA) = 0$, the latter of which implies $A = ABA$ (i.e., $B \in \{A^-\}$).

Note 2.2. Condition A is similar to an *ns* condition for a quadratic form involving a normal random vector to follow a chi-square distribution (e.g., [25,27, Theorem 9.2.1]; [30–32]). There, however, A is *nnd*, and B is symmetric (though not necessarily *nnd*), which obviously does not hold in the present context. There have been extensions of Cochran's theorem to rectangular matrices, however, from a purely algebraic perspective. See Anderson and Styan [1, Theorem 1.2], Baksalary and Hauke [2, Section 2], and Šemrl [29, Section IV] for this line of developments.

Theorem 2.2 (Condition B1). *Let A and B be as defined in Theorem* 2.1. *Then, the following propositions are equivalent*:

(i) *AB is the projector onto* $\operatorname{Sp}(AB)$ *along* $\operatorname{Ker}(AB)$.
(ii) $ABABA = ABA$, *and any one of the following conditions*: (a) $\operatorname{rank}(AB) = \operatorname{rank}(AB)^2$, (b) $\operatorname{rank}(AB) = \operatorname{rank}(ABA)$, (c) $\operatorname{rank}(AB) = \operatorname{rank}(ABABA)$, (d) $\operatorname{rank}(AB) = \operatorname{tr}(AB)$, *and* (e) $\operatorname{rank}(AB) = \operatorname{tr}(AB)^2$.

Proof. As has been remarked in Note 2.1, $\operatorname{rank}(ABA) = \operatorname{rank}(AB)^2 = \operatorname{rank}(ABABA) = \operatorname{tr}(AB) = \operatorname{tr}(AB)^2$ under $ABABA = ABA$, so that Conditions (a) through (e) of (ii) are all equivalent under Condition A. It thus suffices to prove the equivalence of (i) and (ii) for only one of them, say, (a). That (i) implies (ii) is obvious. Conversely, $ABABA = ABA$ implies $(AB)^3 = (AB)^2$, and $\operatorname{rank}(AB)^2 = \operatorname{rank}(AB)$ implies $AB = W(AB)^2$ for some W. Hence, $AB = W(AB)^2 = W(AB)^3 = (AB)^2$. Note that $\operatorname{rank}(AB) = \operatorname{rank}(AB)^2$ is also equivalent to AB being semi-simple, and to $\operatorname{Sp}(AB) \cap \operatorname{Ker}(AB) = \{0\}$ [26, p. 31, Complement 1.9]. □

Condition B1 is stronger than Condition A. The latter will become equivalent to the former if and only if any of the conditions (a) through (e) of (ii) holds.

We can establish a similar condition to B1 for BA.

Corollary 2.1 (Condition B2). *Let A and B be as defined in Theorem* 2.1. *Then*, *the following propositions are equivalent*:

(i) BA *is the projector onto* $\mathrm{Sp}(BA)$ *along* $\mathrm{Ker}(BA)$.
(ii) $ABABA = ABA$ *and any one of the following conditions*: (a) $\mathrm{rank}(BA) = \mathrm{rank}(BA)^2$, (b) $\mathrm{rank}(BA) = \mathrm{rank}(ABA)$, (c) $\mathrm{rank}(BA) = \mathrm{rank}(ABABA)$, (d) $\mathrm{rank}(BA) = \mathrm{tr}(BA)$, *and* (e) $\mathrm{rank}(BA) = \mathrm{tr}(BA)^2$.

The condition in which both $B1$ and $B2$ hold will be called Condition B.

Theorem 2.3. *Let $B_i\,(i = 1, \ldots, m)$ be v by u matrices*, *and let $H = \sum_{i=1}^{m} B_i$. Consider the following conditions*:

(a) $AB_iAB_iA = AB_iA$ *for* $i = 1, \ldots, m$.
(b) $AB_iAB_jA = 0(i \neq j)$ *and* $\mathrm{rank}(AB_iAB_iA) = \mathrm{rank}(AB_iA)$ *for* $i, j = 1, \ldots, m$.
(c) $AHAHA = AHA$.
(d) $\mathrm{rank}(AHA) = \sum_{i=1}^{m} \mathrm{rank}(AB_iA)$.

Then, *any two of the first three conditions imply all other conditions*, *and* (c) *and* (d) *imply* (a) *and* (b).

Proof. We note that (a) $AB_iAB_iA = AB_iA$ if and only if $AB_iAB_iAA^- = AB_iAA^-$, (b) $AB_iAB_jA = 0$ $(i \neq j)$ and $\mathrm{rank}(AB_iAB_iA) = \mathrm{rank}(AB_iA)$ if and only if $AB_iAB_jAA^- = 0$ $(i \neq j)$ and $\mathrm{rank}(AB_iAB_iAA^-) = \mathrm{rank}(AB_iAA^-)$, (c) $AHAHA = AHA$ if and only if $AHAHAA^- = AHAA^-$, and (d) $\mathrm{rank}(AHAA^-) = \sum_{i=1}^{m} \mathrm{rank}(AB_iAA^-)$. Since AB_iAA^- is idempotent, Khatri's [19] Lemma 3 can be directly applied to establish the results in the theorem. See also Anderson and Styan's [1, Theorem 1.2], and Hartwig [13]. □

Note 2.3. As noted in Note 2.2, Condition A is similar to the condition under which a certain quadratic form involving a normal random vector follows a chi-square distribution. Likewise, the conditions stated in Theorem 2.3 resembles those under which two or more quadratic forms involving a normal random vector follow independent chi-square distributions (Cochran's and related theorems; see Rao and Mitra [27, Section 9.3]). A major difference is that in Cochran's and related theorems A is *nnd*, and B_i $(i = 1, \ldots, m)$ are symmetric, whereas in Theorem 2.3 they could both be rectangular.

Note 2.4. Let $H = \sum_{i=1}^{m} B_i$ in Theorem 2.3 satisfy $AHA = A$ (i.e., $H \in \{A^-\}$). Then, the following three propositions, i) $AB_iAB_iA = AB_iA$ for $i = 1, \ldots, m$, (ii) $AB_iAB_jA = 0$ for $i \neq j$ and $i, j = 1, \ldots, m$, and (iii) $\mathrm{rank}(A) = \sum_{i=1}^{m} \mathrm{rank}(AB_iA)$, are equivalent. This can be seen by noting that $AHA = A$ implies Condition (c) of Theorem 2.3. The three propositions above correspond to the three remaining conditions ((a), (b), and (d)) in Theorem 2.3.

We now explicitly assume (5) for B and investigate its consequences.

Note 2.5. Once we assume (5), the following relations hold without any additional conditions.

(a) Both AB and BA have h nonzero eigenvalues which are all unities, and hence $\mathrm{tr}(AB) = \mathrm{tr}(BA) = h = \mathrm{tr}(AB)^2 = \mathrm{tr}(BA)^2$.
(b) (1) $ABABAN = ABAN$ and (2) $(AB)^3 = (AB)^2$, and (1′) $M'ABABA = M'ABA$ and (2′) $(BA)^3 = (BA)^2$.
(c) $M'(A - ABA)N = 0$.

(b) and (c) are trivial, although (a) may require some explanation. Note that AB and $P = (M'AN)^- M'AN$ have the same set of eigenvalues. The latter is idempotent, and consequently it has only unit or zero eigenvalues. The number of unit eigenvalues is equal to $\mathrm{tr}(P) = \mathrm{rank}(P) = \mathrm{rank}(M'AN) = h$. Similarly for BA. Note that while (a), and (2) and (2′) of (b) held only under Condition A earlier, they hold here unconditionally.

Lemma 2.2. *Under the representation of B in* (5), $h = \mathrm{rank}(M'AN) = \mathrm{rank}(ABABA) = \mathrm{rank}(AB)^2 = \mathrm{rank}(BA)^2 = \mathrm{rank}(ABAN) = \mathrm{rank}(M'ABA)$.

Proof. This follows from $\mathrm{rank}(M'AN) \geqslant \mathrm{rank}(ABAN) \geqslant \mathrm{rank}(AB)^2 \geqslant \mathrm{rank}(ABABA) \geqslant \mathrm{rank}(M'ABABAN) = \mathrm{rank}(M'AN)$, and $\mathrm{rank}(M'AN) \geqslant \mathrm{rank}(M'ABA) \geqslant \mathrm{rank}(BA)^2 \geqslant \mathrm{rank}(ABABA) \geqslant \mathrm{rank}(M'ABABAN) = \mathrm{rank}(M'AN)$. □

Theorem 2.4. *Under the representation of B in* (5), *the following equivalences hold*:

(A) *Condition A* $\longleftrightarrow \mathrm{rank}(ABA) = \mathrm{rank}(M'AN)$.
(B) *Condition* $B1 \longleftrightarrow \mathrm{rank}(AB) = \mathrm{rank}(M'AN)$.
(C) *Condition* $B2 \longleftrightarrow \mathrm{rank}(BA) = \mathrm{rank}(M'AN)$.

Proof. (A) Condition A implies $\mathrm{rank}(ABA) = \mathrm{rank}(ABABA)$. We also have $\mathrm{rank}(M'AN) \geqslant \mathrm{rank}(ABAN) \geqslant \mathrm{rank}(ABABA) \geqslant \mathrm{rank}(M'ABABAN) = \mathrm{rank}(M'AN)$, which implies $\mathrm{rank}(M'AN) = \mathrm{rank}(ABABA)$, which in turn implies $\mathrm{rank}(ABA) = \mathrm{rank}(M'AN)$. The converse can be shown as follows. $\mathrm{Rank}(ABA) = \mathrm{rank}(M'AN)$ implies $\mathrm{rank}(ABA) = \mathrm{rank}(ABAN)$, which in turn implies $ABA = ABANW$ for some W. Hence, $ABABA = ABABANW = AN(M'AN)^- M'AN (M'AN)^- M'ANW = ABANW = ABA$. This may also be seen from the fact that under (5), $\mathrm{tr}(AB) = \mathrm{tr}(AB)^2 = h$ is trivially true.

(B) Condition B1 implies $\text{rank}(AB) = \text{rank}(ABAB)$. We also have $\text{rank}(M'AN) \geqslant \text{rank}(ABAN) \geqslant \text{rank}(ABAB) \geqslant \text{rank}(M'ABABAN) = \text{rank}(M'AN)$, which implies $\text{rank}(M'AN) = \text{rank}(ABAB)$, which in turn implies $\text{rank}(AB) = \text{rank}(M'AN)$. The converse can be proven in a manner similar to (A). $\text{Rank}(AB) = \text{rank}(M'AN)$ implies $\text{rank}(AB) = \text{rank}(ABAN)$, which in turn implies $AB = ABANW$ for some W. Hence, $(AB)^2 = ABABANW = AN(M'AN)^- M'AN(M'AN)^- M'ANW = ABANW = AB$.

(C) is similar to (B). □

We now give several other conditions and discuss their relationships to those mentioned above (Conditions A, B1, B2 and B).

Lemma 2.3

(A) *Condition* C1 : *The following propositions are equivalent*:
 (i) $\text{rank}(AN) = \text{rank}(M'AN)$.
 (ii) AB *is the projector onto* $\text{Sp}(AN)$ *along* $\text{Ker}(AB)$.

(B) *Condition* C2: *The following propositions are equivalent*:
 (i) $\text{rank}(M'A) = \text{rank}(M'AN)$.
 (ii) BA *is the projector onto* $\text{Sp}(BA)$ *along* $\text{Ker}(M'A)$.

Proof. See, for example, Theorem 2.1 of Yanai [37]. □

Note that M' is a g-inverse of $AN(M'AN)^-$ under Condition C1, and N is a g-inverse of $(M'AN)^- M'A$ under Condition C2. The condition in which both C1 and C2 are satisfied will be called Condition C.

Theorem 2.5

(A) *Rank Invariance*: $\text{Rank}(AN(M'AN)^- M'A)$ *is invariant over the choice of* $(M'AN)^-$ *if and only if either Condition* C1 *holds or Condition* C2 *holds*.

(B) *Matrix Invariance*: *Matrix* $AN(M'AN)^- M'A$ *is invariant over the choice of* $(M'AN)^-$ *if and only if Condition* C *holds*.

Proof. (A) According to Baksalary and Mathew [4, Theorem 1], for non-null matrices, AN and $M'A$, $\text{rank}(AN(M'AN)^- M'A)$ is invariant over the choice of $(M'AN)^-$ if and only if (a) $\text{Sp}(AN(M'AN)^- M'A)$ is invariant, or (b) $\text{Sp}((AN(M'AN)^- M'A)')$ is invariant. According to Baksalary and Kala [3, Theorem]; see also Groß, [8, Theorem], (a) holds if and only if (c) $\text{Sp}(N'A') \subset \text{Sp}(N'A'M)$ and $\text{Sp}(M'A) \subset \text{Sp}(M'AN)$, or (d) $\text{Sp}(N'A') \subset \text{Sp}(N'A'M)$ and $\text{Sp}(N'A') \cap \text{Sp}(N'A'MQ) = \{0\}$, where Q is a matrix such that $\text{Sp}(Q) = \text{Ker}(A'M)$. We have (e) $\text{Sp}(N'A'M) \subset \text{Sp}(N'A')$ and (f) $\text{Sp}(M'AN) \subset \text{Sp}(M'A)$. Since (f) implies $\text{Sp}(N'A'MQ) = \{0\}$,

(a) holds if and only if $\mathrm{Sp}(N'A) \subset \mathrm{Sp}(M'AN)$, which together with (e) implies $\mathrm{Sp}(N'A') = \mathrm{Sp}(N'A'M)$, or $\mathrm{rank}(AN) = \mathrm{rank}(M'AN)$. Similarly, (b) holds if and only if $\mathrm{rank}(M'A) = \mathrm{rank}(M'AN)$. Whether Condition C1 or C2 holds, the invariant rank of $AN(M'AN)^{-}M'A$ is equal to $\mathrm{rank}(M'AN)$.

(B) directly follows from Rao and Mitra [27, Lemma 2.2.4 (iii) and Complement 2.1]. Matrix $A - AN(M'AN)^{-}M'A$ is invariant if and only if matrix $AN(M'AN)^{-}M'A$ is invariant. □

Lemma 2.4 (Condition D). *The following propositions are equivalent*:

(i) $\mathrm{rank}(B) = \mathrm{rank}(M'AN)$.
(ii) $BAB = B$ (*i.e.*, $A \in \{B^{-}\}$).
(iii) AB *is the projector onto* $\mathrm{Sp}(AB)$ *along* $\mathrm{Ker}(B)$.
(iv) BA *is the projector onto* $\mathrm{Sp}(B)$ *along* $\mathrm{Ker}(BA)$.

Proof. Equivalences among (ii), (iii), and (iv) have been shown by Ben-Israel and Greville [5]. See also (3.16), (3.22), and (3.23) of Cline and Funderlic [7].

Equivalence between (i) and (ii) can be shown as follows: (i) implies $\mathrm{rank}(B)$ is invariant over the choice of $(M'AN)^{-}$, which in turn implies $\mathrm{rank}(N) = \mathrm{rank}(M'AN)$ or $\mathrm{rank}(M) = \mathrm{rank}(M'AN)$. The former implies $N = WM'AN$ for some W. Thus, $BAB = N(M'AN)^{-}M'AN(M'AN)^{-}M' = WM'AN(M'AN)^{-}M'AN(M'AN)^{-}M' = WM'AN(M'AN)^{-}M' = N(M'AN)^{-}M' = B$. The latter implies $M' = M'ANW$ for some W. By a similar argument as above, we obtain $BAB = B$ in this case as well. Conversely, $BAB = B$ implies $(AB)^2 = AB$ (and $(BA)^2 = BA$), so that $\mathrm{rank}(B) = \mathrm{rank}(AB) = \mathrm{rank}(BA) = \mathrm{rank}(BAB) = \mathrm{rank}(AB)^2 = \mathrm{rank}(BA)^2 = \mathrm{rank}(M'AN)$.

Condition (ii) implies that AB is the projector onto $\mathrm{Sp}(AB)$ along $\mathrm{Ker}(AB)$, but $\mathrm{Ker}(B) \subset \mathrm{Ker}(AB) \subset \mathrm{Ker}(BAB) = \mathrm{Ker}(B)$, so that $\mathrm{Ker}(AB) = \mathrm{Ker}(B)$. Conversely, that AB is a projector along $\mathrm{Sp}(B)$ implies $BAB = B$. Condition (ii) also implies that BA is the projector onto $\mathrm{Sp}(BA)$ along $\mathrm{Ker}(BA)$, but $\mathrm{Sp}(B) \supset \mathrm{Sp}(BA) \supset \mathrm{Sp}(BAB) \supset \mathrm{Sp}(B)$, so that $\mathrm{Sp}(BA) = \mathrm{Sp}(B)$. □

Lemma 2.5

(A) *Condition* E1: *The following propositions are equivalent*:
  (i) $\mathrm{rank}(M) = \mathrm{rank}(M'AN)$.
  (ii) AB *is the projector onto* $\mathrm{Sp}(AB)$ *along* $\mathrm{Ker}(M')$.
(B) *Condition* E2: *The following propositions are equivalent*:
  (i) $\mathrm{rank}(N) = \mathrm{rank}(M'AN)$.
  (ii) BA *is the projector onto* $\mathrm{Sp}(N)$ *along* $\mathrm{Ker}(BA)$.

Proof. See, for example, [37]. □

Note that AN is a g-inverse of $(M'AN)^-M'$ under Condition E1, and $M'A$ is a g-inverse of $N(M'AN)^-$ under Condition E2. The condition in which both Conditions E1 and E2 are satisfied is called Condition E. Under this condition B is unique and $BAB = B$ [27, Theorem 4.11.7].

Under Conditions C1 and E1, M' and $AN(M'AN)^-$ are reflexive g-inverses of each other, so are AN and $(M'AN)^-M'$, and AB is the projector onto $\text{Sp}(AN)$ along $\text{Ker}(M')$. Under Conditions C2 and E2, $M'A$ and $N(M'AN)^-$ are reflexive g-inverses of each other, so are N and $(M'AN)^-M'A$, and BA is the projector onto $\text{Sp}(N)$ along $\text{Ker}(M'A)$.

Lemma 2.6 (Condition F). *The following propositions are equivalent*:

(i) $\text{rank}(A) = \text{rank}(M'AN)$.
(ii) $ABA = A$ $(i.e., B \in \{A^-\})$.
(iii) AB *is the projector onto* $\text{Sp}(A)$ *along* $\text{Ker}(AB)$.
(iv) BA *is the projector onto* $\text{Sp}(BA)$ *along* $\text{Ker}(A)$.

Proof. Equivalence between (i) and (ii) follows immediately from Theorem 2.1 of Mitra [22]. (See also the last paragraph of Note 2.1.) Condition (ii) implies AB is the projector onto $\text{Sp}(AB)$ along $\text{Ker}(AB)$, but $\text{Sp}(A) \supset \text{Sp}(AB) \supset \text{Sp}(ABA) = \text{Sp}(A)$, so that $\text{Sp}(AB) = \text{Sp}(A)$. That AB is a projector onto $\text{Sp}(A)$ implies $ABA = A$, establishing the equivalence between (ii) and (iii). (ii) also implies BA is the projector onto $\text{Sp}(BA)$ along $\text{Ker}(BA)$, but $\text{Ker}(A) \subset \text{Ker}(BA) \subset \text{Ker}(ABA) = \text{Ker}(A)$, so that $\text{Ker}(BA) = \text{Ker}(A)$. Conversely, that BA is a projector along $\text{Ker}(A)$ implies $ABA = A$, establishing the equivalence between (ii) and (iv). □

Corollary 2.2

(A) *Condition* $E2 \longrightarrow$ *Condition* $C1 \longrightarrow$ *Condition* $B1 \longrightarrow$ *Condition* A.
(B) *Condition* $E1 \longrightarrow$ *Condition* $C2 \longrightarrow$ *Condition* $B2 \longrightarrow$ *Condition* A.
(C) *Condition* $E1 \longrightarrow$ *Condition* $D \longrightarrow$ *Condition* B, *and Condition* $E2 \longrightarrow$ *Condition* $D \longrightarrow$ *Condition B.*
(D) *Condition* $F \longrightarrow$ *Condition C.*

A proof of this corollary is trivial. Takane and Hunter [33] considered an extension of Wedderburn–Guttman's theorem under Conditions E1 and E2, which are obviously sufficient but not necessary for Condition A.

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